ICASE

THE SPECTRUM OF THE CHEBYSHEV COLLOCATION OPERATOR

FOR THE HEAT EQUATION

David Gottlieb

Liviu Lustman

Report No. 82-12

May 14, 1982

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING

NASA Langley Research Center, Hampton, Virginia 23665

Operated by the

UNIVERSITIES SPACE RESEARCH ASSOCIATION

THE SPECTRUM OF THE CHEBYSHEV COLLOCATION OPERATOR FOR THE HEAT EQUATION

David Gottlieb
Tel-Aviv University
and
Institute for Computer Applications in Science and Engineering

Liviu Lustman
System and Applied Sciences Corporation
Hampton, VA

ABSTRACT

It is shown that the eigenvalues of the pseudospectral Chebyshev second derivative operator with separated boundary conditions are real, negative and distinct. An application to the full potential equation is also presented.

Work was supported in part by the Air Force Office of Scientific Research under Contract No. AFOSR 77-3405 and the National Aeronautics and Space Administration under NASA Contract No. NAS1-15810 while the first author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665. Work for the second author was supported by NASA Contract No. NAS1-1652.

Section 1

Chebyshev pseudospectral methods [3] provide a highly accurate discretization for the general evolution problem

$$\frac{\partial u}{\partial t} = LU. \tag{1.1}$$

Briefly stated, if P_N is the projection operator induced by this discretization method, one solves the equation

$$\frac{\partial u_N}{\partial t} = P_N L P_N U_N, \tag{1.2}$$

which is a finite dimensional equation. Convergence of this method for $0 \leq t \leq T$ is achieved if

$$\lim_{N\to\infty} \| u_N(t) - u(t) \| = 0 \qquad \text{for all} \quad 0 \leq t \leq T, \tag{1.3}$$

and this is assured if

$$\| e^{P_N L P_N t} \| \leq C(t) \qquad 0 \leq t \leq T, \tag{1.4}$$

where $C(t)$ does not depend upon N.

When the solution u eventually reaches a steady state it is important to know whether u_N reaches a steady state. This requires that

$$\lim_{t\to\infty} e^{P_N L P_N t} = 0, \tag{1.5}$$

for fixed N.

It should be noted that equation (1.5) describes a very different limit process than (1.4). In (1.4) we fix t and ask for convergence as the mesh size $1/N$ tends to zero, so for example if $C(t) \sim e^t$ the method is stable. However, in (1.5) we fix N and require $e^{P_N L P_N t}$ to tend to zero as t tends to infinity, so that a $C(t)$ bound in (1.4) that grows in time will not suffice for steady state.

Equation (1.5) is a statement about the spectrum of the matrix $P_N L P_N$ i.e., that the eigenvalues of $P_N L P_N$ have negative real part. Moreover, many numerical methods for the solution of (1.2) require that if the spectrum of L is real so also must be the spectrum of $P_N L P_N$.

In the following we consider the heat equation

$$u_t = u_{xx} \qquad -1 < x < 1 \qquad t > 0, \tag{1.6}$$

with the general boundary conditions

$$\begin{aligned} \alpha u(1,t) + \beta \frac{\partial u}{\partial x}(1,t) &= 0 \\ \gamma u(-1,t) + \delta \frac{\partial u}{\partial x}(-1,t) &= 0, \end{aligned} \tag{1.7}$$

and in Section II we investigate the range of $\alpha,\beta,\gamma,\delta$ for which (1.6) possesses steady state solutions. In Section III we review and modify some conditions that assure that a polynomial has negative, real and distinct roots. The sole purpose of Section III is to provide technical tools for the main result of this paper, i.e., Theorem (4.5) in Section IV.

Section IV contains the main result of this paper. In this section we investigate the pseudospectral Chebyshev method for the problem (1.6) - (1.7) for a subset of the range of the parameters $\alpha,\beta,\gamma,\delta$ discussed in Section II. We find an explicit representation of the characteristic polynomial of the matrix $P_N L P_N$ that corresponds to (1.6) - (1.7) and, by using the theory discussed in Section III, we prove that the roots of this polynomial, i.e., the eigenvalues of the matrix, are real, negative, and distinct.

In Section V we discuss an application of the theory to the full potential equation arising in fluid dynamics. This problem motivated the theoretical problem (1.6) - (1.7) because we found it necessary to patch several computational domains together and to employ Chebyshev expansions in each. This problem can be put in the form (1.6) - (1.7) (for small Mach numbers) and therefore allows a standard iteration technique to be used.

Section II

In this section we investigate the parabolic equation

$$(2.1)\qquad \begin{aligned} &(a) & u_t &= u_{xx} & x &< 1 \\ & & \alpha u(1) + \beta u_x(1) &= 0 & & \\ &(b) & \gamma u(-1) + \delta u_x(-1) &= 0 & \alpha,\gamma &> 0. \end{aligned}$$

In particular we address ourselves to finding those $\alpha,\beta,\gamma,\delta$ for which the solution $u(x,t)$ of (2.1) converges to the steady-state.

Certainly $u(x,t)$ does not converge to steady state for all $\alpha,\beta,\gamma,\delta$, for example, if $\alpha = \beta = \gamma = \delta = 1$ the solution of (2.1) is

$$u(x,t) = e^{t-x},$$

and this solution diverges as $t \to \infty$.

The following theorem gives an algebraic relation between the coefficients $\alpha,\beta,\gamma,\delta$ such that a steady state solution does exist

Theorem (2.1). Let $u(x,t)$ be a solution of (2.1). Then $u(x,t)$ converges as $t \to \infty$ if and only if one of the following conditions holds

(i) $\beta \neq 0$ $\delta \neq 0$ and $2\frac{\alpha\gamma}{\beta\delta} + \frac{\gamma}{\delta} - \frac{\alpha}{\beta} < 0$,

(ii) $\beta = 0$ $\delta \neq 0$ and $\frac{\gamma}{\delta} - 1/2 < 0$,

(iii) $\beta \neq 0$ $\delta = 0$ and $\frac{\alpha}{\beta} + 1/2 > 0$,

(iv) $\beta = 0$ $\delta = 0$.

Proof: 1). First we show that the conditions are sufficient. We proceed by means of an energy estimate: multiply (2.1) by u and integrate to obtain

$$\tag{2.2} \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2 dx = u(1)u_x(1) - u(-1)u_x(-1) - \int_{-1}^{1} u_x^2 dx.$$

However,

$$\tag{2.3} -\int_{-1}^{1} u_x^2 dx < -\frac{1}{2}\,[u(1)-u(-1)]^2,$$

and therefore

$$\tag{2.4} \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2 dx < u(1)u_x(1) - u(-1)u_x(-1) - \frac{1}{2}\,[u(1)-u(-1)]^2.$$

This is an estimate for the L_2 norm

$$\|u\|_2 \overset{\text{def}}{\equiv} \left(\int_{-1}^{1} u^2(x,t)dx\right)^{1/2}$$

As a function of time, the norm is non-increasing whenever the right-hand expression in (2.4) is negative or zero.

If condition (iv) of the theorem holds then $u(1) = u(-1) = 0$ so that

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2 dx < 0,$$

and therefore the norm does not increase.

For condition (i), $\beta \neq 0 \neq \delta$. Incorporating (2.1b) into (2.4) one gets

$$\tag{2.5} \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2 dx < -\left(\frac{\alpha}{\beta}+\frac{1}{2}\right)u^2(1) + u(1)u(-1) + \left(\frac{\gamma}{\delta}-\frac{1}{2}\right)u^2(-1).$$

The right hand side of (2.5) will be non-positive if

$$2\,\frac{\alpha\gamma}{\beta\delta} + \frac{\gamma}{\delta} - \frac{\alpha}{\beta} < 0,$$

in which case, again, $\frac{d}{dt}\|u\|_2^2 < 0$. Suppose now that $\beta = 0$,

$\delta \neq 0$ as in case (ii); then (2.4) becomes

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2 dx \leq \left(\frac{\gamma}{\delta} - \frac{1}{2}\right)u^2(-1),$$

and therefore the condition

$$\frac{\gamma}{\delta} - \frac{1}{2} \leq 0,$$

implies a non increasing norm. Similarly, case (iii) implies if $\gamma = 0$, $\beta \neq 0$, so that

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2 dx = -\left(\frac{\alpha}{\beta} + \frac{1}{2}\right)u^2(1) \leq 0.$$

Consider now separation of variables for (1.6):

$$\text{(2.6)} \qquad \begin{cases} V = A + Bx \\ V = e^{\lambda^2 t \pm \lambda x} \end{cases}$$

- with A, B and complex $\lambda \neq 0$ determined by boundary conditions. By explicit check, we find:

a. if $\|V\|_2$ is constant in time, then $V(x,t) = \text{constant}$;

b. if $\|V\|_2$ decreases, then $v(x,t)$ decays exponentially to zero with t.

Since the general solution of (1.6) - (1.7) is a superposition of the functions (2.6), we see that (i) - (iv) are sufficient for convergence to steady state, as they imply

$$\|u\|_2(t) \leq \|u\|_2(0).$$

2). We now establish the necessity of conditions (i) - (iv) of the Theorem. Consider a solution of the form

$$w = e^{\lambda^2 t}\left(Ae^{\lambda x}+Be^{-\lambda x}\right).$$

The function w satisfies (2.1a). In order for w to satisfy (2.1b) for nontrivial A,B, λ must satisfy the following determinental equation

$$f(\lambda) = e^{2\lambda}(\gamma-\lambda\delta)(\alpha+\beta\lambda) - e^{-2\lambda}(\alpha-\beta\lambda)(\gamma+\lambda\delta) = 0. \tag{2.7}$$

One solution is $\lambda = 0$; then $A = -B$ so that $w \equiv 0$. If (2.7) allows a real non-zero solution λ then there is a function $w(x,t)$ which satisfies (2.1) and grows in time. We show that if none of (i-iv) are satisfied real λ will exist, and thus establish the necessity part of the theorem. Suppose first that $\beta \neq 0 \neq \delta$; then for λ large enough sign $f(x) = -\text{sign}(\beta\delta)$. Let ε be positive and small enough such that

$$\text{sign } f(\varepsilon) = \text{sign}\left[f(0) + \varepsilon f'(0)\right]. \tag{2.8}$$

Since

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 4\alpha\gamma + 2\gamma\beta - 2\alpha\delta, \end{aligned} \tag{2.9}$$

then

$$\text{sign } f(\varepsilon) = \text{sign}(\beta\delta)\ \text{sign}\left(\frac{\alpha\gamma}{\beta\delta} + \frac{\gamma}{\delta} - \frac{\alpha}{\beta}\right),$$

and therefore if condition (i) is violated and if

$$\frac{\alpha\gamma}{\beta\delta} + \frac{\gamma}{\delta} - \frac{\alpha}{\beta} > 0,$$

then for λ sufficiently large

$$\text{sign } f(\varepsilon) = \text{sign}(\beta\delta) = -\ \text{sign } f(\lambda);$$

hence there is a λ_0 real such that $f(\lambda_0) = 0$ and we can find a solution

$w(x,t) = e^{\lambda_0^2 t}\left(Ae^{\lambda_0 x} + Be^{-\lambda_0 x}\right)$ that increases in time.

If now $\beta = 0 \neq \delta$ then for large λ

$$\text{sign } f(x) = -\text{sign } \alpha\delta = -\text{sign } \delta.$$

From (2.8) $f'(0) = \alpha(2\gamma-\delta)$ and therefore

$$\text{sign } f(\varepsilon) = \text{sign } \delta \text{ sign}\left(\frac{\gamma}{\delta} - 1/2\right),$$

so that if $\frac{\gamma}{\delta} - 1/2 > 0$ then

$$\text{sign } f(\varepsilon) = -\text{sign } f(\lambda)$$

for large λ, again demonstrating the existence of a real λ_0 satisfying (2.6). A similar argument holds if $\beta \neq 0 = \delta$.

Hence, if none of the conditions (i) - (iv) of the theorem are satisfied no steady state solution exists as $t \to \infty$. This demonstrates the necessity of (i) - (iv) and the theorem is proved. Note that the theorem is a statement of the non-negativity of the operator $-\frac{\partial^2}{\partial x^2}$ with the boundary conditions (2.1b).

Section III

This section discusses some aspects of the theory of the location of zeroes of real polynomials. We shall review a few classical conditions ensuring that a real polynomial has real, negative and distinct roots. These theorems will be used in the next section in connection with pseudospectral Chebyshev methods.

Throughout this section $\Omega_1(\nu)$ and $\Omega_2(\nu)$ will be real polynomials of degree m and $(m-1)$ (or m) respectively.

<u>Definition 3.1.</u> $\Omega_1(\nu)$ and $\Omega_2(\nu)$ form a positive pair if: a) the roots $\nu_1,\cdots,\nu_m$ of Ω_1 and the roots $\nu'_1,\cdots,\nu'_{m-1}$ (or $\nu'_1,\cdots,\nu'_m$) of Ω_2 are all distinct real and negative; b) the roots alternate as follows

$$\nu_1 < \nu'_1 < \cdots < \nu'_{m-1} < \nu_m < 0 \tag{3.1}$$

$$(\text{or} \quad \nu'_1 < \nu_1 < \nu'_2 \cdots < \nu'_m < \nu_m < 0);$$

(c) the highest coefficients of Ω_1 and Ω_2 are of like sign.

Lemma 3.1 characterizes such positive pairs.

<u>Lemma 3.1:</u> The polynomial

$$h(z) = \Omega_1(z^2) + z\Omega_2(z^2), \tag{3.2}$$

is a Hurwitz polynomial (i.e., all of its roots have negative real parts) if and only if $\Omega_1(\nu)$ and $\Omega_2(\nu)$ form a positive pair.
For the proof of this lemma see [1], p. 228.

<u>Lemma 3.2:</u> Let p and q be positive numbers, let Ω_1 and Ω_2 form a positive pair and

$$u(\nu) = p\Omega_1(\nu) + q\Omega_2(\nu), \tag{3.3}$$

then the roots of $u(\nu)$ are real, negative, and, distinct.

<u>Proof:</u> By Lemma 3.1

$$h(z) = \Omega_1(z^2) + z\Omega_2(z^2)$$

is a Hurwitz polynomial. Therefore $g(z)$ defined by

$$g(z) = (q + zp)h(z)$$

is also Hurwitz. But

$$g(z) = [q\Omega_1(z^2)+pz^2\Omega_2(z^2)] + z[p\Omega_1(z^2)+q\Omega_2(z^2)],$$

and by Lemma 3.1 $p\Omega_1(\nu) + q\Omega_2(\nu)$ and $q\Omega_1(\nu) + p\nu\Omega_2(\nu)$ form a positive pair.

In particular the roots of

$$u(\nu) = p\Omega_1(\nu) + q\Omega_2(\nu)$$

are real, negative, and distinct, which proves the lemma.

We are now ready for the two main results of this section.

<u>Lemma 3.3</u>: Let p, q, r, s be positive numbers and let $\Omega_1(\nu)$ and $\Omega_2(\nu)$ be a positive pair of polynomials. Define $u(\nu)$ and $v(\nu)$ by

$$\begin{aligned} u(\nu) &= p\Omega_1(\nu) + q\Omega_2(\nu) \\ v(\nu) &= r\Omega_1(\nu) + s\Omega_2(\nu). \end{aligned} \tag{3.4}$$

Then $u(\nu)$ and $v(\nu)$ form a positive pair.

<u>Proof</u>: By Lemma 3.2 the roots of $u(\nu)$ and $v(\nu)$ are real, negative, and distinct. It remains to prove that they interlace.

From the discussion in [1], p. 227 it is clear that if $g(\nu)$ and $h(\nu)$ form a positive pair then $pg + qh$ and g form a positive pair. Therefore $(p/q)\Omega_1 + \Omega_2$ and $(r/s)\Omega_1$ form a positive pair. Suppose now that $(r/s) > (p/q)$ and define $h = (p/q)\Omega_1 + \Omega_2$ and $g = \big((r/s)-(p/q)\big)\Omega_1$; then clearly $h+g$ and h form a positive pair. This completes the proof.

Lemma 3.4. Let $\Omega_1(\nu)$, $\Omega_2(\nu)$ and $\theta_1(\nu),\theta_2(\nu)$ be two positive pairs, let

$$h(\nu) = \Omega_1\theta_2 + \Omega_2\theta_1.$$

Then the roots of $h(\nu)$ are real negative and distinct.

Proof: By Lemma 3.1 the polynomials $h(z)$ and $g(z)$ defined by

$$h(z) = \Omega_1(z^2) + z\Omega_2(z^2),$$
$$g(z) = \theta_1(z^2) + z\theta_2(z^2)$$

are Hurwitz polynomials. Therefore $h(z) \cdot g(z)$ is a Hurwitz polynomial. But, since

$$h(z)g(z) = [\Omega_1(z^2)\theta_1(z^2)+z^2\Omega_2(z^2)\theta_2(z^2)] + z[\Omega_1(z^2)\theta_2(z^2)+\Omega_2(z^2)\theta_1(z^2)],$$

Lemma 3.1 implies that the polynomials

$$\Omega_1(\nu)\theta_1(\nu) + z^2\Omega_2(\nu)\theta_2(\nu) \quad \text{and} \quad \Omega_1(\nu)\theta_2(\nu) + \Omega_2(\nu)\theta_1(\nu),$$

form a positive pair and in particular their roots are real, negative, and distinct.

Section IV

The pseudospectral Chebyshev method for space discretization of (2.1) involves seeking a polynomial $u_N(x,t)$ of degree N in x, such that

$$\text{(a)} \qquad \frac{\partial u_N}{\partial t} - \frac{\partial^2 u_N}{\partial x^2} = 0 \qquad \text{for } x = x_j = \cos\frac{\pi j}{N} \qquad j=1,\cdots,N-1$$

(4.1) and

$$\text{(b)} \qquad \begin{aligned} &\alpha u_N(x_0,t) + \beta \frac{\partial u_N}{\partial x}(x_0,t) = 0 \\ &\gamma u_N(x_N,t) + \delta \frac{\partial u_N}{\partial x}(x_N,t) = 0. \end{aligned}$$

We refer the reader to [2], [3] for a discussion of an efficient implementation of (4.1).

Suppose now that $u_N(x,t) = e^{\lambda t}\phi_N(x,\lambda)$; then

$$\text{(a)} \qquad \lambda\phi_N - \frac{\partial^2 \phi_N}{\partial x^2} = 0 \qquad \text{for } x = x_j \quad j=1,\cdots,N-1$$

(4.2) and

$$\text{(b)} \qquad \begin{aligned} &\alpha\phi_N(1,\lambda) + \beta \frac{\partial \phi_N}{\partial x}(1,\lambda) = 0 \\ &\gamma\phi_N(-1,\lambda) + \delta \frac{\partial \phi_N}{\partial x}(-1,\lambda) = 0. \end{aligned}$$

In this section we will prove that for $\beta\delta \leq 0$ the possible eigenvalues λ are real, negative, and distinct. Hence (4.1) is amenable to standard iteration techniques. Define $E_N(x)$ by

$$E_N(x) = \lambda\phi_N(x,\lambda) - \frac{\partial^2 \phi_N}{\partial x^2}(x,\lambda).$$

$E_N(x)$ is a polynomial of degree N in x; moreover by (4.2a)

$$E_N(x_j) = 0 \qquad j=1,\cdots,N-1. \tag{4.3}$$

Let now $T_N(x)$ be the Nth order Chebyshev polynomial namely

$$T_N(x) = \cos(N\cos^{-1}x).$$

The points x_j, $j=1,\cdots,N-1$ are the extrema of $T_N(x)$ in the open interval $-1 < x < 1$ and therefore

$$T_N'(x_j) = 0. \tag{4.4}$$

Since E_N and T_N' share the zeros x_j, $0 < j < N$, while their respective degrees are N and $N-1$, there exist values A, B independent of x such that:

$$E_N(x) = (A+Bx)T_N'(x),$$

and therefore

(4.5) $$\lambda\phi_N(x,\lambda) - \frac{\partial^2\phi_N}{\partial x^2}(x,\lambda) = (A+Bx)T_N'(x).$$

where A and B are determined by (4.2b).

Equation (4.5) can be solved explicitly as follows:

Lemma 4.1. Let $\psi_N(x,\lambda)$ and $\chi_N(x,\lambda)$ defined by

(4.6)

(a) $$\psi_N(x,\lambda) = \sum_{k=0}^{\infty} \lambda^{-k-1} \frac{\partial^{2k+1}}{\partial x^{2k+1}} T_N(x)$$

(b) $$\chi_N(x,\lambda) = \sum_{k=0}^{\infty} \lambda^{-k-1} \frac{\partial^{2k}}{\partial x^{2k}} \left(x \frac{\partial T_N}{\partial x}(x)\right).$$

Then

(4.7) $$\phi_N(x,\lambda) = A\psi_N(x,\lambda) + B\chi_N(x,\lambda).$$

Proof. First, note that the solution of the homogeneous equation

$$\lambda\phi_N(x,\lambda) - \frac{\partial^2\phi_N}{\partial x^2}(x,\lambda) = 0,$$

is not a polynomial in x and therefore the only solution of (4.5) which is a polynomial is

$$-(D^2-\lambda)^{-1}(A+Bx)T_N',$$

where $D = \partial/\partial x$. This completes the proof.

Note that upon defining $\nu = \lambda^{-1}$, $\psi_N(x, \frac{1}{\nu})$ and $\chi_N(x,\frac{1}{\nu})$ are polynomials both in x and in ν. If $N = 2M$, $\psi_N(x,\lambda)$ is of order M in ν and $\chi_N(x,\lambda)$ is of order $M+1$ in ν. For simplicity we will assume that $N = 2M$; for odd N we only need to redefine ψ_N and χ_N to get the proper degrees in ν and reach the same conclusions. Note also that if $N = 2M$

$$\psi_N(x,\lambda) = -\psi_N(-x,\lambda)$$

(4.8) and

$$\chi_N(x,\lambda) = \chi_N(-x,\lambda).$$

Substituting (4.7) in (4.2) we get

(4.9)

$$A\left[\alpha\psi_N(\psi,\lambda)+\beta \frac{\partial\psi_N}{\partial x}(1,\lambda)\right] + B\left[\alpha\chi_N(1,\lambda)+\beta \frac{\partial\chi_N}{\partial x}(1,\lambda)\right] = 0$$

$$A\left[\gamma\psi_N(-1,\lambda)+\delta \frac{\partial\psi_N}{\partial x}(-1,\lambda)\right] + B\left[\gamma\chi_N(-1,\lambda)+\delta \frac{\partial\chi_N}{\partial x}(-1,\lambda)\right] = 0.$$

In order to have a nontrivial solution for A and B the determinant of the coefficients in (4.9) must vanish, i.e.,

$$\left[\alpha\psi_N(1,\lambda) + \beta \frac{\partial\psi_N}{\partial x}(1,\lambda)\right]\left[\gamma\chi_N(-1,\lambda) + \delta \frac{\partial\chi_N}{\partial x}(-1,\lambda)\right]$$

$$- \left[\alpha\chi_N(1,\lambda)+\beta \frac{\partial\chi_N}{\partial x}(1,\lambda)\right]\left[\gamma\psi_N(-1,\lambda)+\delta \frac{\partial\psi_N}{\partial x}(-1,\lambda)\right] = 0,$$

and by (4.8) we arrive at the following characteristic equation

(4.10)

$$0 = \left[\alpha\psi_N(1,\lambda)+\beta \frac{\partial\psi_N}{\partial x}(1,\lambda)\right]\left[\gamma\chi_N(1,\lambda)-\delta \frac{\partial\chi_N}{\partial x}(+1,\lambda)\right]$$

$$+ \left[\alpha\chi_N(1,\lambda)+\beta \frac{\partial x_N}{\partial x}(1,\lambda)\right]\left[\gamma\psi_N(1,\lambda)-\delta \frac{\partial\psi_N}{\partial x}(1,\lambda)\right].$$

Note that the right-hand side of (4.10) is a polynomial of degree $2N+1$ in ν; two of its roots are $\nu_1 = \nu_2 = 0$ by (4.6). The lemma we are about to prove will show that the other roots are real, negative, and distinct. To establish this we need the following:

Lemma 4.2. Let

$$f_N(x,\mu) = \sum_{k=0}^{\infty} \mu^{k+1} \frac{\partial^{k+1}}{\partial x^{k+1}} T_N(x), \tag{4.11}$$

and

$$g_N(x,\mu) = \sum_{u=0}^{\infty} \mu^{k+1} \frac{\partial^{k}}{\partial x^{k}} \left(x \frac{\partial}{\partial x} T_N(x)\right). \tag{4.12}$$

Then $(1/\mu)f_N(1,\mu)$ and $(1/\mu)g_N(1,\mu)$ are Hurwitz polymials, that is, their roots have negative real parts.

Proof: From (4.11) and (4.12)

$$\frac{1}{\mu} f_N(x,\mu) - \frac{\partial}{\partial x} f_N(x,\mu) = T_N'(x).$$

Now let $w_N = e^{(1/\mu)t} f_N(x,\mu)$ then

$$\frac{\partial w_N}{\partial t} - \frac{\partial w_N}{\partial x} = e^{(1/\mu)} T_N'(x), \tag{4.13}$$

and if μ is a root of $(1/\mu)f_N(1,\mu)$ then

$$w_N(1,t) = 0. \tag{4.14}$$

In [2] it has been shown that $w_N(x,t)$ decreases in time and therefore real $\mu < 0$.

A similar argument holds for (4.12).

Lemma 4.2 is essential for proving:

Lemma 4.3. Let

$$\Omega_1(\nu) = \frac{1}{\nu}\,\psi_N\left(1,\frac{1}{\nu}\right)$$

$$\Omega_2(\nu) = \frac{1}{\nu}\,\frac{\partial \psi_N}{\partial x}\left(1,\frac{1}{\nu}\right).$$

Then $\Omega_1(\nu)$ and $\Omega_2(\nu)$ form a positive pair.

Proof: Construct

$$h(\mu) = \Omega_1(\mu^2) + \mu\Omega_2(\mu^2),$$

$$h(\mu) = \sum_{k=0}^{\infty} \mu^{2k} \frac{\partial^{2k+1}}{\partial x^{2k+1}} T_N(1) + \sum_{k=0}^{\infty} \mu^{2k+1} \frac{\partial^{2k+2}}{\partial x^{2k+2}} T_N(1)$$

$$= \frac{1}{\mu} \sum_{k=0}^{\infty} \mu^{k+1} \frac{\partial^{k+1}}{\partial x^{k+1}} T_N(1) = \frac{1}{\mu} f_N(1,\mu).$$

By Lemma 4.2, $h(\mu)$ is a Hurwitz polynomial and therefore by Lemma 3.1, $\Omega_1(\nu)$ and $\Omega_2(\nu)$ form a positive pair.

Lemma 4.4. Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta < 0$. Then

$$\frac{1}{\nu}\left[\alpha\psi_N\left(1,\frac{1}{\nu}\right) + \beta\,\frac{\partial\psi_N}{\partial x}\left(1,\frac{1}{\nu}\right)\right]$$

and

$$\frac{1}{\nu}\left[\gamma\psi_N\left(1,\frac{1}{\nu}\right) - \delta\,\frac{\partial\psi_N}{\partial x}\left(1,\frac{1}{\nu}\right)\right]$$

form a positive pair.

Proof: The proof is a direct consequence of Lemmas 3.3 and 4.3 by identifying

$$p = \alpha, \quad q = \beta, \quad r = \gamma, \quad s = -\delta.$$

The results of Lemma 4.3 and 4.4 hold if we replace ϕ_N by χ_N.

We are now in a position to state the main result of this paper:

Theorem 4.5. Let $\alpha, \beta, \gamma > 0$ and $\delta < 0$; then the solutions λ of (4.2) - i.e., the eigenvalues of the Chebyshev second derivatives operator with boundary conditions, are real, negative, and distinct.

Proof: First of all, zero cannot be an eigenvalue. By substituting $\lambda = 0$ into (4.5), we reach

$$\frac{\partial^2 \phi_N}{\partial x^2} = (A + Bx)T_N'(x).$$

The second derivative is a polynomial of degree N-2 or less, which vanishes at the N-1 distinct zeros of T_N', and thus vanishes identically. Therefore:

$$\phi_N = p + qx.$$

It is easy to check that no constants p and q, not both zero, can be found to satisfy the boundary conditions, if $\alpha,\beta,\gamma > 0$ and $\delta < 0$. As the solutions of (4.2) also satisfy (4.10), we may consider only the nonzero roots of (4.10) as possible eigenvalues. Let

$$\nu\Omega_1(\nu) = \alpha\psi_N\left(1,\tfrac{1}{\nu}\right) + \beta \frac{\partial\psi_N}{\partial x}\left(x,\tfrac{1}{\nu}\right),$$

$$\nu\Omega_2(\nu) = \gamma\psi_N\left(1,\tfrac{1}{\nu}\right) - \delta \frac{\partial\psi_N}{\partial x}\left(x,\tfrac{1}{\nu}\right),$$

$$\nu\theta_1(\nu) = \alpha\chi_N\left(1,\tfrac{1}{\nu}\right) + \beta \frac{\partial\chi_N}{\partial x}\left(1,\tfrac{1}{\nu}\right),$$

$$\nu\theta_2(\nu) = \gamma\chi_N\left(1,\tfrac{1}{\nu}\right) - \delta \frac{\partial\chi_N}{\partial x}\left(1,\tfrac{1}{\nu}\right),$$

by Lemma 4.4 $\Omega_1(\nu)$ and $\Omega_2(\nu)$ form a positive pair and $\theta_1(\nu)$ and $\theta_2(\nu)$ form a positive pair. Equation (4.10) can be written as

$$0 = \nu^2[\Omega_1(\nu)\theta_2(\nu) + \Omega_2(\nu)\theta_1(\nu)],$$

and by Lemma 3.4 the roots $\nu_1,\cdots,\nu_{2n-1}$ must be real, negative, and distinct. This proves the theorem.

Note: The conditions of Theorem 4.5 can be relaxed to include the following cases

1) $\gamma > 0$, $\delta < 0$,
2) $\alpha = 0$ or $\beta = 0$.

However, the results do not quite cover the cases included in Theorem 2.1, for example the case $\alpha,\beta,\gamma,\delta > 0$, $\frac{\gamma}{\delta} < 1/2$.

Section V

In many applications of spectral methods it is preferable to divide the computational domain into several subdomains and apply the pseudospectral Chebyshev method for each domain [4]. The solutions in the subdomains are then required to satisfy certain continuity relations. It is the purpose of this section to show that this procedure gives rise to a numerical operator with real, negative, and distinct eigenvalues.

Consider the model problem

$$(5.1)\qquad \begin{aligned} w_t &= w_{xx} \qquad -1 < x < 3 \\ w(-1) &= w(3) = 0. \end{aligned}$$

We divide the domain into two subdomains, $-1 < x < 1$ and $1 < x < 3$, and apply the pseudospectral method to each subdomain. Thus two polynomials u_N, v_N of degree N are constructed satisfying

$$(5.2)\qquad \begin{aligned} \frac{\partial u_N}{\partial t} &= \frac{\partial^2 u_N}{\partial x^2} \qquad x = x_j = \cos\frac{\pi j}{N} \qquad j=1,\cdots,N-1 \\ \frac{\partial v_N}{\partial t} &= \frac{\partial^2 v_N}{\partial x^2} \qquad x = y_j = 2 + \cos\frac{\pi j}{N} \qquad j=1,\cdots,N-1 \end{aligned}$$

$$u_N(-1) = v_N(3) = 0,$$

and the continuity equations:

$$(5.3)\qquad \begin{aligned} &\text{(a)} \qquad u_N(1,t) = v_N(1,t) \\ &\text{(b)} \qquad \frac{\partial u_N}{\partial x}(1,t) = \frac{\partial v_N}{\partial x}(1,t). \end{aligned}$$

Equation (5.3) is implemented by setting $u_N(1,t) = v_N(1,t) = f(t)$, where $f(t)$ satisfies (5.3b) in the collocation sense.

Now define

$$v_N(x,t) \equiv v_N(2-x,t),$$

to get:

$$\left.\begin{aligned} \frac{\partial u_N}{\partial t} &= \frac{\partial^2 u_N}{\partial x^2} \\ \frac{\partial v_N}{\partial t} &= \frac{\partial^2 v_N}{\partial x^2} \end{aligned}\right\} \quad \text{at } x = x_j, \tag{5.4}$$

while (5.3) becomes:

$$\begin{aligned} u_N(1,t) &= v_N(1,t) \\ \frac{\partial v_N}{\partial x}(1,t) &= -\frac{\partial v_N}{\partial x}(1,t). \end{aligned} \tag{5.5}$$

Consider

$$\begin{aligned} R(x,t) &= u_N(x,t) + v_N(x,t) \\ S(x,t) &= u_N(x,t) - v_N(x,t). \end{aligned} \tag{5.6}$$

These satisfy the differential equations:

$$\left.\begin{aligned} \frac{\partial R}{\partial t} &= \frac{\partial^2 R}{\partial x^2} \\ \frac{\partial S}{\partial t} &= \frac{\partial^2 S}{\partial x^2}, \end{aligned}\right. \quad \text{at } x = x_j \tag{5.7}$$

with boundary conditions

$$(5.8)\qquad \begin{aligned} &(a) \quad R(1,t) = R(-1,t) = 0 \\ &(b) \quad \frac{\partial S}{\partial x}(1,t) = S(-1,t) = 0. \end{aligned}$$

From the discussion in Section IV, it follows that the eigenvalues of (5.7) - (5.8) are real, negative and distinct, and the problem is therefore amenable to standard iteration techniques.

An application to the above analysis has been made to the numerical solution of the full potential equation. The flow around a wing is described in terms of a potential Φ which satisfies

$$(5.9)\qquad \frac{\partial}{\partial x}\left(\rho \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \Phi}{\partial y}\right) = 0.$$

The density ρ is obtained in terms of Φ by

$$(5.10)\qquad \rho = \left[1 - M_\infty^2 \frac{\gamma-1}{2}\left(\Phi_x^2+\Phi_y^2-1\right)\right]^{\frac{1}{\gamma-1}}.$$

The wing is represented by the "small disturbance" boundary conditions

$$(5.11)\qquad \Phi_y = \frac{df}{dx}\Phi_x$$

on the wing, where $y = f(x)$ is the geometric description of the wing.

We have used three computational domains: the middle one represents the wing while the others contain regions of free flow (see Figure 2). The use of three domains bunches grid points at the wing tips and allows increased accuracy. Had we used one computational domain, the middle segment, representing the wing would have occupied a reign of coarse mesh, producing poor resolution.

We solve (5.9) by modifying it to

$$(5.12)\qquad \frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \Phi_y\right),$$

and using the DuFort-Frankel algorithm for (5.12).
On the rest of the boundary (not including the wing) we specified

$$\frac{\partial \Phi}{\partial y} = 0 \qquad \text{on } AB,\ CD$$

(5.13)

$$\Phi = q_\infty x \qquad \text{on } DEFGHA,$$

(see Figure 2) and

$$\Phi,\ \Phi_x \ \text{ continuous across } BG,\ CF.$$

It is essential for the stability of the DuFort-Frankel algorithm that the eigenvalues are real and negative, a fact that is established above.

A typical flow over a parabolic wing

$$f(x) = \tau(1-x^2) \qquad -1 < x < 1,$$

for $M_\infty = .5$ and $\tau = .2$ is presented in Figure 3.

REFERENCES

[1] F. R. GANTMACHER, The Theory of Matrices, Chelsea Publishing Company, New York, NY 1974.

[2] D. GOTTLIEB, "The Stability of Pseudospectral Chebyshev Methods," Math. Comp., Vol. 36, No. 153, January 1981, pp. 107-118.

[3] D. GOTTLIEB and S. A. ORSZAG, Numerical Analysis of Spectral Methods: Theory and Application, NSF-CBMS Regional Conference Series in Applied Mathematics 26, SIAM, Philadelphia, PA 1977.

[4] S. A. ORSZAG, "Spectral Methods for Problems in Complex Geometries," J. Computational Phys., Vol 37, 1980, pp. 70-92.

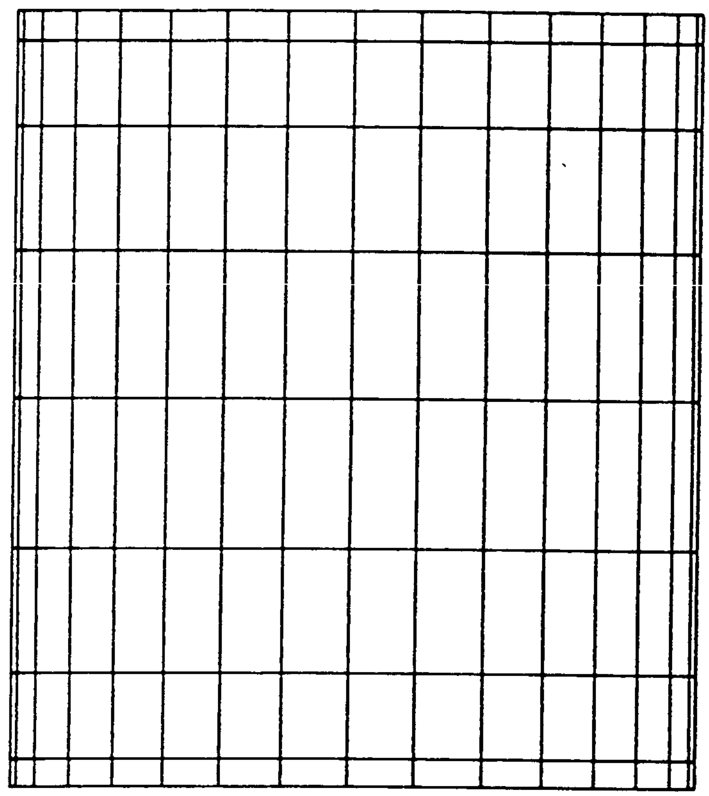

Figure 1. A Chebyshev mesh 17 x 9

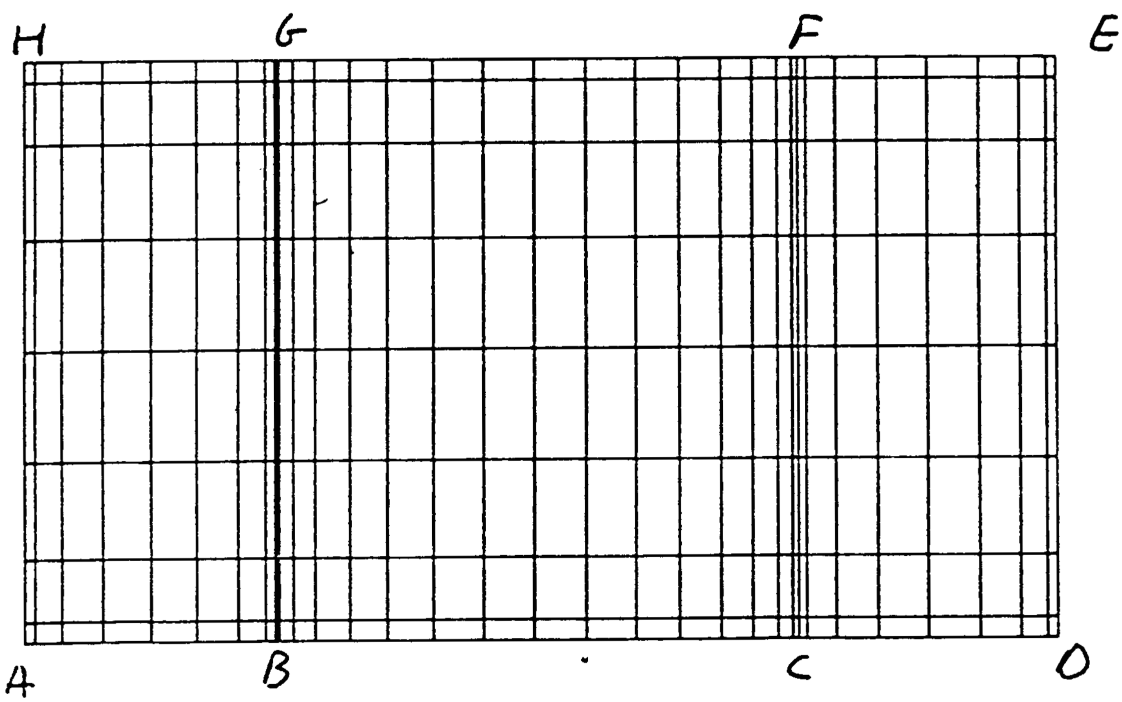

Figure 2. Three Chebyshev meshes. BC represents the wing.

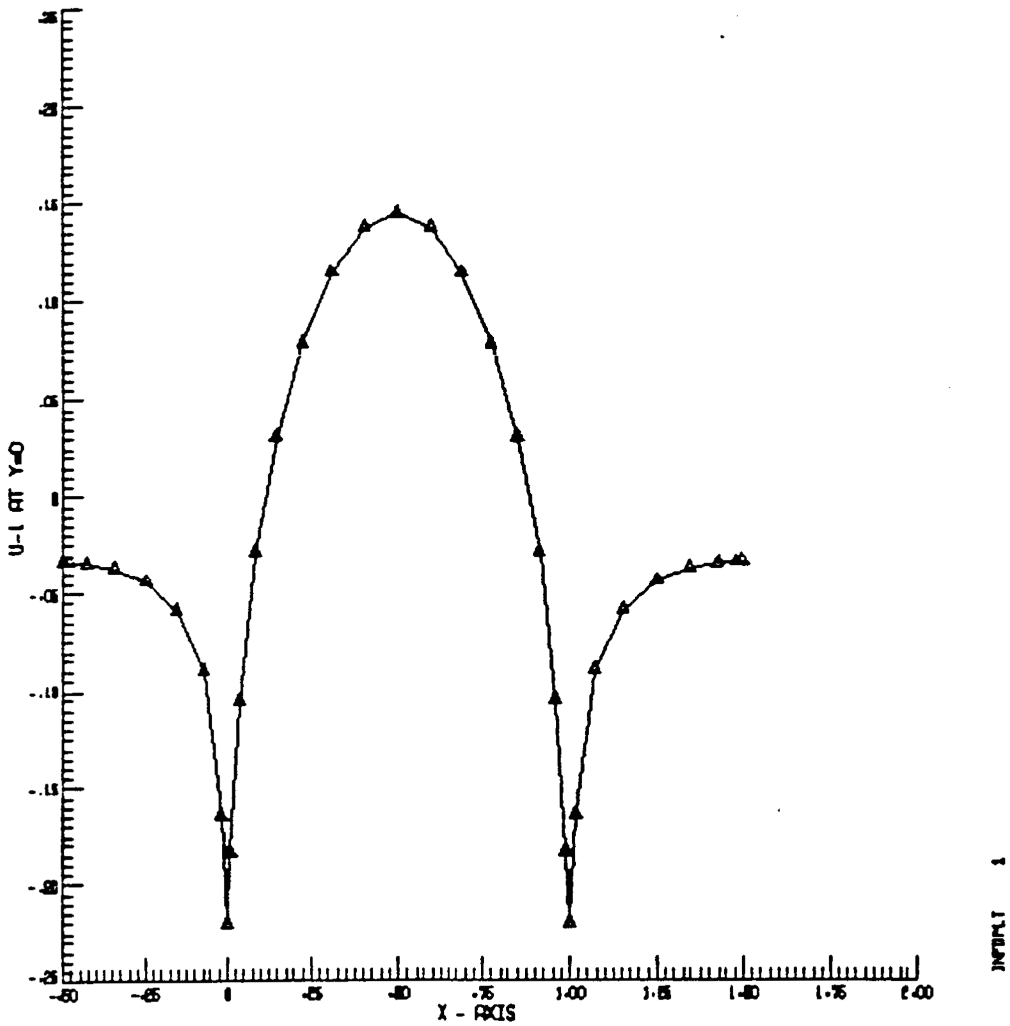

Figure 3. Velocity profile on the wing.